Descriptive Complexity of Graph Spectra

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Outline

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Getting Started

The goal would be to generalise in a syntactical way the property of being a graph uniquely determined by its spectrum.

Getting Started

A **graph** is an element of the elementary class axiomatised by the first-order sentence

$$\forall x \forall y (\neg E(x,x) \wedge (E(x,y) \rightarrow E(y,x)),$$

where E is a binary relation symbol interpreted as an irreflexive symmetric binary relation called adjacency.

Getting Started

The *adjacency matrix* A of a graph G is the 01 matrix with rows and columns indexed by the vertices of G, such that the ij-entry of A is equal to 1 if i, j are adjacent vertices, and 0 otherwise.

Definition

The **spectrum** of a graph is the multi-set of eigenvalues of its adjacency matrix.

Even though it is defined in terms of the adjacency matrix of G, the spectrum does not depend on the order in which the vertices of G are listed.

$$sp(G) = \{\lambda \in \mathbb{R} : det(\lambda I - A) = 0\}$$

Getting Started

Suppose G is a graph with adjacency matrix A.

Definition

A **walk** of length ℓ in G is a sequence $(v_0, v_1, \ldots, v_\ell)$ of vertices of G, such that consecutive vertices are adjacent in G.

We say that a given walk of length ℓ is *closed* if $v_0 = v_\ell$.

For each $\ell > 0$, the number of walks of length ℓ in G starting at i and ending at j is given by

$$(A^\ell)_{ij}$$

and the number of closed walks of length ℓ in G is given by

$$tr(A^\ell).$$

Outline

Cospectral Graphs

Definition

Two graphs with the same spectrum are called **cospectral**.

Two adjacency matrices A, B have the same spectrum if there is an orthogonal matrix which conjugates A into B.

Recall that the rows and columns of any orthogonal matrix are orthonormal vectors. So if A, B are adjacency matrices of two cospectral graphs, there exists U with $U^T U = UU^T = I$ such that

$$U^T AU = B.$$

Cospectral Graphs

Proposition

The total number of closed walks in two cospectral graphs coincides for each length.

Proof.

$$tr(A) = tr(AUU^T) = tr(U^T AU) = tr(B).$$

Outline

Counting Logic

C^k denotes the set of FO-formulas augmented by a *counting quantifier* using only k variables:

$$\exists^i x \varphi.$$

Counting Logic

Proposition

C^3-equivalence implies cospectrality.

Proof. There exists a C^3-sentence φ_k^ℓ such that

$$G \models \varphi_k^\ell$$

if and only if

$$tr(A^\ell) = k.$$

Counting Logic

Proposition

Cospectrality does not imply C^3-equivalence.

Proof.

$$sp(K_1 \cup C_4) = sp(K_{1,4})$$

however,

$$K_{1,4} \not\models \exists x \forall y \neg E(x, y).$$

Infinitary Finite Variable Logic

Lemma

There exist pairs of cospectral graphs which are not $L^k_{\infty,\omega}$-equivalent for all $k \geq 2$.

Proof.

$$K_1 \cup C_4 \quad \text{and} \quad K_{1,4}.$$

Infinitary Finite Variable Logic

Lemma

For each $k \geq 2$, there exist pairs of $L^k_{\infty,\omega}$-equivalent graphs which are not cospectral.

Proof.

$$P(q_k) \quad \text{and} \quad P^3(q_k)$$

with

$$q_k > k^2 2^{4k}.$$

Infinitary Finite Variable Logic

Proposition

For cospectrality and $L^k_{\infty,\omega}$-equivalence neither implies the other for all $k \geq 2$.

Outline

DS Graphs

An equivalence relation R on graphs (such as cospectrality) spans a graph class $\mathcal{R}$ in the following sense. A graph G is in $\mathcal{R}$ if all graphs H which are in the same equivalence class as G wrt R are isomorphic to G:

$$\mathcal{R} = \{G : \forall H \ (G \equiv_R H \Rightarrow G \cong H)\}.$$

DS Graphs

Definition

Let DS be the graph class spanned by cospectrality and say a graph is **determined by its spectrum** if it is in the class DS.

This means that G is in DS iff it is the only graph up to isomorphism that has $sp(G)$ as spectrum.

Haemers' Conjecture

Conjecture (Dam & Haemers, 2003)

The following statement is true:

$$\lim_{n\to\infty} \frac{|\mathrm{DS} \cap \mathcal{G}_n|}{|\mathcal{G}_n|} = 1.$$

Outline

Fixed Point Logic

Formation rule: If φ is a formula positive in the relation variable R and with free variables from $\mathbf{x}$, then

$$[\mathbf{lfp}_{R,\mathbf{x}}\varphi](\mathbf{a})$$

is also a formula.

By closing FO under this formation rule we obtain the *fixed-point logic* FP.

Fixed Point Logic

The semantics for $[\mathbf{lfp}_{R,\mathbf{x}}\varphi](\mathbf{a})$ are defined as follows: the tuple $\mathbf{a}$ of elements from a structure $\mathfrak{A}$ is in the least fixed point of the operator that maps R to $\varphi(R, \mathbf{x})$.

The least fixed point of the mapping $R \mapsto \varphi(R, \mathbf{x})$ is the limit of the sequence:

$$\begin{aligned} R^0 &= \emptyset \\ R^{m+1} &= \{\mathbf{a} : (\mathfrak{A}, R^m) \models \varphi[\mathbf{a}/\mathbf{x}]\}. \end{aligned}$$

Fixed Point Logic

On structures which come equipped with a linear order FP expresses exactly the classes that are decidable in polynomial time.

Partial Fixed Point Logic

For any formula $\varphi(R, \mathbf{x})$ and a structure $\mathfrak{A}$, the sequence

$$
\begin{aligned}
R^0 &= \emptyset \\
R^{m+1} &= \{\mathbf{a} : (\mathfrak{A}, R^m) \models \varphi[\mathbf{a}/\mathbf{x}]\}
\end{aligned}
$$

may not converge to a fixed point.

The partial fixed point of $R \mapsto \varphi(R, \mathbf{x})$ is the limit of this sequence if exists, and $\emptyset$ otherwise.

Partial Fixed Point Logic

The *partial fixed-point logic* PFP is the set of formulae obtained by closing FO under the formula formation rule:

$$[\mathbf{pfp}_{R,\mathbf{x}}\varphi](\mathbf{a}).$$

Partial Fixed Point Logic

Over the class of orderer structures PFP expresses exactly the classes that are decidable in polynomial space.

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Outline

Fixed Point Logics with Counting

The *fixed-point logic with counting* FPC and the *partial fixed-point logic with counting* PFPC are extensions of two-sorted FO with the ability to express the cardinality of definable sets via counting terms:

$$\#_x \varphi.$$

Fixed Point Logics with Counting

The intended semantics of counting terms is that $\#_x\varphi$ denotes the number (i.e. the element of $\{0, ..., n\}$) of elements that satisfy the formula φ.

Outline

Definability

Any PSPACE property of ordered structures is definable in PFPC.

Similarly, FPC can express any polynomial-time decidable property of ordered structures.

Cospectrality

We can construct a single FPC formula $\varphi(\lambda, \kappa_1, \kappa_2)$ that is satisfied by a graph G iff the number of closed walks of length λ in G is k and (κ_1, κ_2) encode this k.

An obstacle here was that k can be exponentially-sized and thus cannot be expressed as tuple of number variables. Instead we use a binary relation on the number domain.

Cospectrality

Proposition

Cospectrality is definable in FPC.

DS Property

The property of being in the graph class DS can be decided in Π_2^P and therefore is in PSPACE.

DS Property

In the absence of a linear order, FPC and PFPC are strictly weaker than the complexity classes P and PSPACE respectively.

DS Property

Proposition

The class of graphs that are DS is definable in PFPC.

Proof.

$$\begin{aligned}\forall\alpha\forall\beta\mathbf{pfp}_{\bar{R},\mu,\nu,\kappa}[&(\forall\mu\nu\bar{R}(\mu,\nu,0))\wedge\theta(\bar{R})\wedge\kappa=0\vee\\ &\vee\neg\theta(\bar{R})\wedge\kappa\neq 0\vee\\ &\vee\theta(\bar{R})\wedge\text{next}(\bar{R},\mu,\nu)\wedge\kappa=0](\alpha,\beta,0).\end{aligned}$$

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Conclusions

We have shown that any pair of graphs that are elementarily equivalent with respect to the three-variable counting first-order logic C^3 are cospectral, and this is not the case with C^2, nor with any number of variables if we exclude counting quantifiers.

Cospectrality is a property of graphs that does not follows from any finite collection of extension axioms, or equivalently, from any first-order sentence with asymptotic probability 1.

We have also shown that cospectrality is definable in fixed-point logic with counting and the class of graphs that are determined by their spectra is definable in partial fixed-point logic with counting.

Conclusions

The proportion of graphs that are determined up to isomorphism by their L^k theory tends to 0. On the other hand, it is known that almost all graphs are determined by their C^2 theory and so by their C^3 theory.

We have established that cospectrality is incomparable with L^k-equivalence for any k; is incomparable with C^2-equivalence; and is subsumed by C^3-equivalence.

Our results are compatible with either answer to the open question of whether almost all graphs are DS.

Thanks.